11-5-21

One last Spherical Example

Ex: compute the volume of closed disk of radius $\alpha > 0$

NB: done in cartesian coordinates but it was nasty

$$D_\alpha = \{(\rho, \theta, \varphi) : 0 \le \rho \le \alpha,\ 0 \le \theta \le 2\pi,\ 0 \le \varphi \le \pi\}$$

$$\mathrm{Vol}(D_\alpha) = \iiint_{D_\alpha \, \text{cart}} 1 \, dV$$

$$= \iiint_{D_\alpha \, \text{spher}} 1 \cdot \rho^2 \sin(\varphi) \, dV_{\text{sphere}}$$

$$= \int_{\rho=0}^{\alpha} \int_{\theta=0}^{2\pi} \int_{\varphi=0}^{\pi} \rho^2 \sin(\varphi) \, d\varphi \, d\theta \, d\rho$$

$$= \int_{\rho=0}^{\alpha} \int_{\theta=0}^{2\pi} -\rho^2 [\cos\varphi]_{\varphi=0}^{\pi} \, d\theta \, d\rho$$

$$= \int_{\rho=0}^{\alpha} \int_{\theta=0}^{2\pi} -\rho^2(-1-1) \, d\theta \, d\rho$$

$$= 2 \int_{\rho=0}^{\alpha} \int_{\theta=0}^{2\pi} \rho^2 \, d\theta \, d\rho = 2 \int_{\rho=0}^{\alpha} \rho^2 [\theta]_{\theta=0}^{2\pi} \, d\rho = 4\pi \int_{\rho=0}^{\alpha} \rho^2 \, d\rho$$

$$= \frac{4}{3}\pi(\alpha^3 - 0) = \frac{4}{3}\pi \alpha^3$$

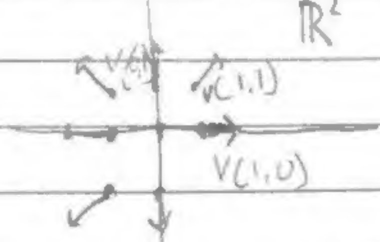
§16.1: Vector Fields

Goal: Study functions $\vec{V}: \mathbb{R}^n \to \mathbb{R}^n$
(most of time $n=2$ or $n=3$)

Vector field - is a function $\vec{v}: \mathbb{R}^n \to \mathbb{R}^n$

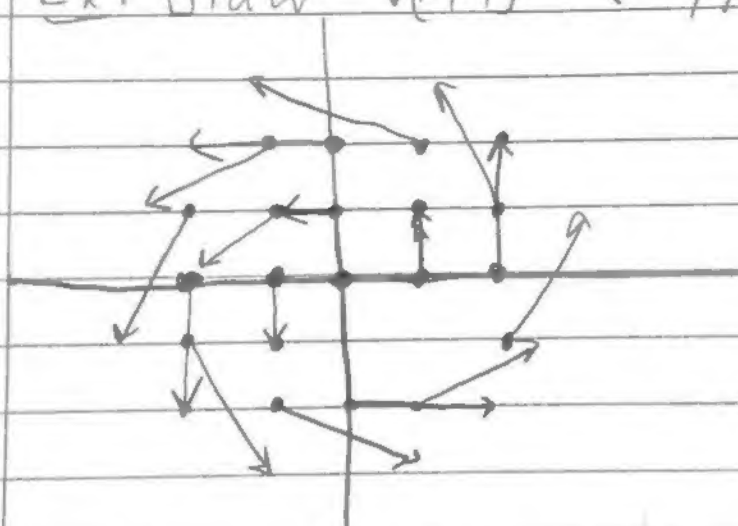
$\vec{V}(x, y) = \langle x, y \rangle$ on $\mathbb{R}^2$

every point in $\mathbb{R}^2$ has an associated vector attatched to it by this vector field.



Ex: Draw $\vec{v}(x, y) = \langle -y, x \rangle$

$V(0,0) = \langle 0,0 \rangle$
$V(1,0) = \langle 0,1 \rangle$
$V(2,0) = \langle 0,2 \rangle$
$V(-1,0) = \langle 0,-1 \rangle$
$V(-2,0) = \langle 0,-2 \rangle$
$V(-1,1) = \langle -1,-1 \rangle$
$V(0,1) = \langle -1,0 \rangle$
$V(-2,1) = \langle -1,-2 \rangle$



Ex: The gradient of a function is always a vector field
↳ e.g. for $f(x,y) = xy$ $\nabla f = \langle y, x \rangle$ is a vector field on $\mathbb{R}^2$. Vector field = "vf"

A vector field like this is sometimes called a "gradient vector field

e.g. $f(x,y,z) = e^{x+y^2}\cos(z+x)$

$\nabla f = \langle e^{x+y^2}\cos(z+x) - e^{x+y^2}\sin(z+x),\ 2y\cos(z+x)e^{x+y^2},\ -e^{x+y^2}\sin(z+x)\rangle$

is a vector field $\vec{\nabla} f(x,y,z) =$

Obvious Question: How do we know when a vector field is a gradient vector field?

↳ is every v.f. a grad v.f.?

Terminology: A conservative vector field is a gradient v.f.

If $\vec{V} = \nabla f$ is a conservative v.f., we call f a potential function for $\vec{V}$

Now, is every v.f. conservative?

On $\mathbb{R}^2$, a conservative v.f. has form $\vec{v} = \langle f_x(x,y), f_y(x,y)\rangle$ for some potential function f. By Clairaut's Theorem, $f_{xy} = f_{yx}$, so every conservative v.f. $\vec{J} = \langle \alpha(x,y), \beta(x,y)\rangle$ has to satisfy

$$\alpha_y = \beta_x$$

Ex: $\vec{V} = \langle \overset{V_x}{-y}, \overset{V_y}{x} \rangle$

$\frac{\partial}{\partial y}[V_x] = \frac{\partial}{\partial y}[-y] = -1$ $\quad$ $\vec{V}$ is non conservative it violates Clairaut's Theorem

$\frac{\partial}{\partial x}[V_y] = \frac{\partial}{\partial y}[x] = 1$

$\therefore$ Not every vector field is a gradient vector field !!

Prop: A vector field $\vec{v} = \langle V_{x_1}, V_{x_2}, \dots, V_{x_n} \rangle$ is conservative if and only if $\frac{\partial}{\partial x_i}\left[V_{x_j}\right] = \frac{\partial}{\partial x_j}\left[V_{x_i}\right]$ for all i, j (i.e. a vf is conservative if and only if it satisfies Clairaut's theorem)

Ex: Is $\vec{v} = \langle x, y \rangle$ conservative?

Sol: $\frac{\partial V_x}{\partial y} = \frac{\partial}{\partial y}[x] = 0$

$\frac{\partial V_y}{\partial x} = \frac{\partial}{\partial x}[y] = 0$

$\therefore$ by the proposition $\vec{v} = \langle x, y \rangle$ is conservative what is it's potential function?

By conservativity, $\nabla f = \vec{v}$ for function $f(x,y)$ i.e. $f_x(x,y) = x$ and $f_y(x,y) = y$.

B/c $\frac{\partial f}{\partial x} = x$, we know $f(x,y) = \int \frac{\partial f}{\partial x}\,dx = \int x\,dx = \frac{1}{2}x^2 + C(y)$

$$\therefore\ y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left[\frac{1}{2}x^2 + C(y)\right] = \frac{dC}{dy}$$

honest constant

$$\therefore\ C(y) = \int \frac{dC}{dy}\,dy = \int y\,dy = \frac{1}{2}y^2 + D$$

$f(x,y) = \frac{1}{2}x^2 + C(y) = \frac{1}{2}x^2 + \frac{1}{2}y^2 + D$ for any constant

D is a potential function of $\vec{v}$.

Ex: $\vec{v} = \langle 2xy, x^2 - 3y^2 \rangle$ conservative?
if yes, potential?

Sol:

$$\frac{\partial V_y}{\partial x} = \frac{\partial}{\partial x}\left[x^2 - 3y^2\right] = 2x$$

$$\frac{\partial V_x}{\partial y} = \frac{\partial}{\partial y}\left[2xy\right] = 2x$$

$\vec{v}$ is conserved

$$f = \int f_x\,dx = \int 2xy\,dx = x^2 y + C(y)$$

$f_y = \frac{\partial}{\partial y}\left[x^2 y + C(y)\right]$ D → honest constant

$x^2 - 3y^2 = x^2 + C'(y)$

$C'(y) = -3y^2$ $C(y) = \int -3y^2\,dy = -y^3 + D$

$f(x,y) = x^2 y + C(y) = x^2 y - y^3 + D$

NB: the method for computing the potentials can be used to prove the proposition from earlier...

Ex: $\vec{v} = \langle \ln(x+y), e^{x+y} + \frac{1}{x+y} \rangle$
conservative? If yes, potential?

Sol: $\frac{\partial v_x}{\partial y} = \frac{\partial}{\partial y}\left[\ln(x+y)\right] = \frac{1}{x+y}$ $\cancel{\times}$

$\frac{\partial v_y}{\partial x} = \frac{\partial}{\partial x}\left[e^{x+y} + \frac{1}{x+y}\right] e^{x+y} - (x+y)^{-2}$

Not conservative. No potential ▨

Last time: you saw

$$\int_{y=c}^{d}\int_{x=a}^{b} f(x)\, g(y)\, dx\, dy = \int_{x=a}^{b} f(x)dx \cdot \int_{y=c}^{d} g(y)dy$$

true only when
① integrating over rectangle $[a,b] \times [c,d]$
② Integrad is a "separable function"
i.e. $h(x,y) = f(x) \cdot g(y)$